GRADIENT BOUNDS FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

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To Pr. David Nualart 60th's birthday

ABSTRACT. We study some functional inequalities satisfied by the distribution of the solution of a stochastic differential equation driven by fractional Brownian motions. Such functional inequalities are obtained through new integration by parts formulas on the path space of a fractional Brownian motion.

CONTENTS

1. INTRODUCTION

Let $(X_t^x)_{t\geq 0}$ be the solution of a stochastic differential equation

$$X_t = x + \sum_{i=1}^{n} \int_0^t V_i(X_s^x) dB_s^i,$$

where $(B_t)_{t\geq 0}$ is a n-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Under ellipticity assumptions and classical boundedness conditions (see [3] and [11]), the random variable X_t^x, $t > 0$, admits a smooth density with respect to the Lebesgue measure of $\mathbb{R}^n$ and the functional operator

$$P_t f(x) = \mathbb{E}\left(f(X_t^x)\right)$$

is regularizing in the sense that it transforms a bounded Borel function f into a smooth function $P_t f$ for $t > 0$. In this note we aim to quantify precisely this regularization property and prove that, under the above assumptions, bounds of the type:

$$|V_{i_1}\cdots V_{i_k} P_t f(x)| \leq C_{i_1\cdots i_k}(t,x)\|f\|_\infty, \quad t>0, x\in\mathbb{R}^n,$$

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are satisfied. We are moreover able to get an explicit blow up rate when $t \to 0$: For a fixed $x \in \mathbb{R}^n$, when $t \to 0$,

$$C_{i_1 \cdots i_k}(t,x) = O\left(\frac{1}{t^{kH}}\right).$$

Our strategy to prove such bounds is the following. If f is a C^∞ bounded function on $\mathbb{R}^n$, we first prove (see Lemma 4.1) that the following commutation holds

$$V_i P_t f(x) = \mathbb{E}\left(\sum_{k=1}^n \alpha_k^i(t,x) V_k f(X_t^x)\right),$$

where the $\alpha(t,x)$'s solve an explicit system of stochastic differential equations. Then, using an integration by parts formula in the path space of the underlying fractional Brownian motion (see Theorem Theorem 3.6) we may rewrite the expectation of the right hand side of the above inequality as $\mathbb{E}\left(\Phi_i(t,x) f(X_t^x)\right)$ where $\Phi_i(t,x)$ is shown to be bounded in L^p, $1 \le p < +\infty$ with a blow up rate that may be controlled when $t \to 0$. It yields a bounds on $|V_i P_t f(x)|$. Bounds on higher order derivatives are obtained in a similar way, by iterating the procedure just described. Let us mention here that the bounds we obtain depends on L^p bounds for the inverse of the Malliavin matrix of X_t^x. As of today, to the knowledge of the authors such bounds have not yet been obtained in the rough case $H < \frac{1}{2}$. The extension of our results to the case $H < \frac{1}{2}$ is thus not straightforward.

We close the paper by an interesting geometric situation where we may prove an optimal and global gradient bound with a constant that is independent from the starting point x. In the situation where the equation is driven by a Brownian motion such global gradient bound is usually related to lower bounds on the Ricci curvature of the Riemannian geometry given by the vector fields V_i's, which makes interesting the fact that the bound also holds with fractional Brownian motions.

2. Stochastic differential equations driven by fractional Brownian motions

We consider the Wiener space of continuous paths:

$$\mathbb{W}^n = (\mathcal{C}([0,1],\mathbb{R}^n), (\mathcal{B}_t)_{0 \le t \le 1}, \mathbb{P})$$

where:

(1) $\mathcal{C}([0,1],\mathbb{R}^n)$ is the space of continuous functions $[0,1] \to \mathbb{R}^n$;
(2) $(\beta_t)_{t \ge 0}$ is the coordinate process defined by $\beta_t(f) = f(t)$, $f \in \mathcal{C}([0,1],\mathbb{R}^n)$;
(3) $\mathbb{P}$ is the Wiener measure;
(4) $(\mathcal{B}_t)_{0 \le t \le 1}$ is the ($\mathbb{P}$-completed) natural filtration of $(\beta_t)_{0 \le t \le 1}$.

A n-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a Gaussian process

$$B_t = (B_t^1, \dots, B_t^n), \ t \ge 0,$$

where $B^1, \dots, B^n$ are n independent centered Gaussian processes with covariance function

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

It can be shown that such a process admits a continuous version whose paths are Hölder γ continuous, $\gamma < H$. Throughout this paper, we will always consider the 'regular' case,

$H > 1/2$. In this case the fractional Brownian motion can be constructed on the Wiener space by a Volterra type representation (see [5]). Namely, under the Wiener measure, the process

$$B_t = \int_0^t K_H(t,s) d\beta_s, t \geq 0 \tag{2.1}$$

is a fractional Brownian motion with Hurst parameter H, where

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \,, \qquad t > s.$$

and c_H is a suitable constant.

Denote by $\mathcal{E}$ the set of step functions on $[0,1]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

The isometry K_H^* from $\mathcal{H}$ to $L^2([0,1])$ is given by

$$(K_H^* \varphi)(s) = \int_s^1 \varphi(t) \frac{\partial K_H}{\partial t}(t,s) dt.$$

Moreover, for any $\varphi \in L^2([0,1])$ we have

$$\int_0^1 \varphi(s) dB_s = \int_0^1 (K_H^* \varphi)(s) d\beta_s.$$

Let us consider for $x \in \mathbb{R}^n$ the solution $(X_t^x)_{t \geq 0}$ of the stochastic differential equation:

$$X_t^x = x + \sum_{i=1}^n \int_0^t V_i(X_s^x) dB_s^i, \tag{2.2}$$

where the V_i's are C^∞ bounded vector fields in $\mathbb{R}^n$. Existence and uniqueness of solutions for such equations have widely been studied and are known to hold in this framework (see for instance [10]). Moreover, the following bounds were proved by Hu and Nualart as an application of fractional calculus methods.

Lemma 2.1. *(Hu-Nualart, [7]) Consider the stochastic differential equation (2.2). If the derivatives of V_i's are bounded and Hölder continuous of order $\lambda > 1/H - 1$, then*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t|^p \right) < \infty$$

for all $p \geq 2$. If furthermore V_i's are bounded and $\mathbb{E}(\exp(\lambda |X_0|^q)) < \infty$ for any $\lambda > 0$ and $q < 2H$, then

$$\mathbb{E}\left(\exp \lambda \left(\sup_{0 \leq t \leq T} |X_t|^q \right) \right) < \infty$$

for any $\lambda > 0$ and $q < 2H$.

Throughout our discussion, we assume that the following assumption is in force:

Hypothesis 2.2.
(1) $V_i(x)$'s are bounded smooth vector fields on $\mathbb{R}^n$ with bounded derivatives at any order.
(2) For every $x \in \mathbb{R}^n$, $(V_1(x), \cdots, V_n(x))$ is a basis of $\mathbb{R}^n$.

Therefore, in this framework, we can find functions ω_{ij}^k such that

$$[V_i, V_j] = \sum_{k=1}^n \omega_{ij}^k V_k, \tag{2.3}$$

where the ω_{ij}^k's are bounded smooth functions on $\mathbb{R}^n$ with bounded derivatives at any order.

3. Integration by parts formulas

We first introduce notations and basic relations for the purpose of our discussion. Consider the diffeomorphism $\Phi(t,x) = X_t^x : \mathbb{R}^n \to \mathbb{R}^n$. Denote by $\mathbf{J_t} = \frac{\partial X_t^x}{\partial x}$ the Jacobian of $\Phi(t,\cdot)$. It is standard (see [11] for details) that

$$d\mathbf{J}_t = \sum_{i=1}^n \partial V_i(X_t^x)\mathbf{J}_t dB_t^i, \qquad \text{with } \mathbf{J}_0 = \mathbf{I}, \tag{3.4}$$

and

$$d\mathbf{J}_t^{-1} = -\sum_{i=1}^n \mathbf{J}_t^{-1}\partial V_i(X_t^x) dB_t^i, \qquad \text{with } \mathbf{J}_0^{-1} = \mathbf{I}. \tag{3.5}$$

For any C_b^∞ vector field W on $\mathbb{R}^n$, we have that

$$(\Phi_{t*}W)(X_t^x) = \mathbf{J}_t W(x), \qquad \text{and} \quad (\Phi_{t*}^{-1}W)(x) = \mathbf{J}_t^{-1}W(X_t^x).$$

Here Φ_{t*} is the push-forward operator with respect to the diffeomorphism $\Phi(t,x) : \mathbb{R}^n \to \mathbb{R}^n$. Introduce the non-degenerate $n \times n$ matrix value process

$$\alpha(t,x) = (\alpha_j^i(t,x))_{i,j=1}^n \tag{3.6}$$

by

$$(\Phi_{t*}V_i)(X_t^x) = \mathbf{J}_t(V_i(x)) = \sum_{k=1}^n \alpha_k^i(t,x)V_k(X_t^x) \qquad i = 1,2,...,n.$$

Note that $\alpha(t,x)$ is non-degenerate since we assume V_i's form a basis at each point $x \in \mathbb{R}^n$. Denote by

$$\beta(t,x) = \alpha^{-1}(t,x). \tag{3.7}$$

Clearly we have

$$(\Phi_{t*}^{-1}V_i(X_t^x))(x) = \mathbf{J}_t^{-1}V_i(X_t^x)(x) = \sum_{k=1}^n \beta_k^i(t,x)V_k(x) \qquad i = 1,2,...,n. \tag{3.8}$$

Lemma 3.1. *Let $\alpha(t,x)$ and $\beta(t,x)$ be as above, we have*

$$d\alpha_j^i(t,x) = -\sum_{k,l=1}^n \alpha_k^i(t,x)\omega_{lk}^j(X_t^x)dB_t^l, \qquad \text{with} \quad \alpha_j^i(0,x) = \delta_j^i; \tag{3.9}$$

and

$$d\beta_j^i(t,x) = \sum_{k,l=1}^n \omega_{li}^k(X_t^x)\beta_j^k(t,x)dB_t^l, \qquad \text{with} \qquad \beta_j^i(0,x) = \delta_j^i. \tag{3.10}$$

Proof. The initial values are apparent by the definition of α and β. We show how to derive equation (3.10). Once the equation for $\beta(t,x)$ is obtained, it is standard to obtain that $\alpha(t,x)=\beta^{-1}(t,x)$.
Consider the $n\times n$ matrix $V=(V_1,V_2,...,V_n)=(V_j^i)$ obtained from the vector fields V. Let W be the inverse matrix of V. It is not hard to see we have

$$\beta_j^i(t,x)=\sum_{k=1}^n W_k^j(x)(\mathbf{J}_t^{-1}V_i(X_t^x))^k(x).$$

By the equation for X_t^x, relation (2.3), equation (3.5), and Itô's formula, we obtain

$$\begin{aligned} d(\mathbf{J}_t^{-1}V_i(X_t^x))(x) &= \sum_{k=1}^n (\mathbf{J}_t^{-1}[V_k,V_i](X_t^x))(x)dB_t^k \\ &= \sum_{k,l=1}^n \omega_{ki}^l(X_t^x)(\mathbf{J}_t^{-1}V_l(X_t^x))(x)dB_t^k. \end{aligned}$$

Hence

$$d\beta_j^i(t,x)=\sum_{k,l=1}^n \omega_{ki}^l(X_t^x)\beta_j^l(t,x)dB_t^k.$$

This completes our proof. □

Define now $h_i(t,x):[0,1]\times\mathbb{R}^n\to\mathcal{H}$ by

$$h_i(t,x)=(\beta_i^k(s,x)\mathbb{I}_{[0,t]}(s))_{k=1,...,n},\qquad i=1,...,n. \tag{3.11}$$

Introduce $M_{i,j}(t,x)$ given by

$$M_{i,j}(t,x)=\frac{1}{t^{2H}}\langle h_i(t,x),h_j(t,x)\rangle_{\mathcal{H}}. \tag{3.12}$$

For each $t\in[0,1]$, consider the semi-norms

$$\|f\|_{\gamma,t}:=\sup_{0\le v<u\le t}\frac{|f(u)-f(v)|}{(u-v)^\gamma}.$$

The semi-norm $\|f\|_{\gamma,1}$ will simply be denoted by $\|f\|_\gamma$.
We have the following two important estimates.

Lemma 3.2. *Let $\alpha(t,x)$, $\beta(t,x)$ and $h_i(t,x)$ be as above. We have:*
(1) For any multi-index ν, integers $k,p\ge 1$, there exists a constant $C_{k,p}(x)>0$ depending on k,p and x such that for all $x\in\mathbb{R}^n$

$$\sup_{0\le t\le 1}\left\|\frac{\partial^{|\nu|}}{\partial x^\nu}\alpha(t,x)\right\|_{k,p}<C_{k,p}(x),\quad \sup_{0\le t\le 1}\left\|\frac{\partial^{|\nu|}}{\partial x^\nu}\beta(t,x)\right\|_{k,p}<C_{k,p}(x).$$

(2) For all integers $k,p\ge 1$, $\delta h_i(t,x)\in\mathbb{D}^{k,p}$. Moreover, there exists a constant $C_{k,p}(x)$ depending on k,p and x such that

$$\|\delta h_i(t,x)\|_{k,p}<C_{k,p}(x)t^H,\qquad t\in[0,1].$$

In the above δ is the adjoint operator of $\mathbf{D}$.

Proof. The result in (1) follows from equation (3.9), (3.10) and Lemma 2.1. In what follows, we show (2). Note that we have (c.f. Nualart[9])

$$\begin{aligned}\delta h_i(t,x) &= \int_0^1 h_i(t,x)_u dB_u - \alpha_H \int_0^1\int_0^1 \mathbf{D}_u h(t,x)_v |u-v|^{2H-2} dudv \\ &= \int_0^t \beta_i(u,x) dB_u - \alpha_H \int_0^t\int_0^t \mathbf{D}_u \beta_i(v,x) |u-v|^{2H-2} dudv.\end{aligned}$$

Here $\alpha_H = H(2H-1)$. From the above representation of δh_i and the result in (1), it follows immediately that $\delta h_i(t,x) \in \mathbb{D}^{k,p}$ for all integers $k,p \geq 1$. To show

$$\|\delta h_i(t,x)\|_{k,p} < C_{k,p}(x) t^H \qquad \text{for all } t \in [0,1],$$

it suffices to prove

$$\left|\int_0^t \beta_i(u,x) dB_u\right| \leq C(x) t^H \qquad t \in [0,1].$$

Here $C(x)$ is a random variable in $L^p(\mathbb{P})$. Indeed, by standard estimate, we have

$$\left\|\int_0^{\cdot} (\beta_i(u,x) - \beta_i(0,x)) dB_u\right\|_{\gamma,t} \leq C\|\beta(\cdot,x)\|_{\tau,t}\|B\|_{\gamma,t}, \qquad t \in [0,1].$$

In the above $\frac{1}{2} < \tau, \gamma < H$ and $\tau + \gamma > 1$, and $C > 0$ is a constant only depending on γ. Therefore

$$\left|\int_0^t \beta_i(u,x) dB_u\right| \leq C\|\beta(\cdot,x)\|_{\tau,t}\|B\|_{\gamma,t} t^\gamma + |\beta(0,x)||B_t|, \qquad t \in [0,1].$$

Together with the fact that for any $\tau < H$, there exists a random variable $G_\tau(x)$ in $L^p(\mathbb{P})$ for all $p > 1$ such that

$$|\beta(t,x) - \beta(s,x)| < G_\tau(x)|t-s|^\tau,$$

the proof is now completed. $\square$

Lemma 3.3. *Let $M(t,x) = (M_{i,j}(t,x))$ be given in (3.12). We have for all $p \geq 1$,*

$$\sup_{t\in[0,1]} \mathbb{E}\left[\det(M(t,x))^{-p}\right] < \infty.$$

Proof. Denote the Malliavin matrix of X_t^x by $\Gamma(t,x)$. By definition

$$\Gamma_{i,j}(t,x) = \langle \mathbf{D}_s X_t^i, \mathbf{D}_s X_t^j \rangle_{\mathcal{H}} = \alpha_H \int_0^t\int_0^t \mathbf{D}_u X_t^i \mathbf{D}_v X_t^j |u-v|^{2H-2} dudv.$$

It can be shown that for all $p > 1$ (cf. Baudoin-Hairer [3], Hu-Nualart [7] and Nualart-Saussereau [11]),

$$\sup_{t\in[0,1]} \mathbb{E}\left(\det\frac{\Gamma(t,x)}{t^{2H}}\right)^{-p} < \infty. \tag{3.13}$$

Introduce γ by

$$\gamma_{i,j}(t,x) = \alpha_H \int_0^t\int_0^t \sum_{k=1}^n \left(\mathbf{J}_u^{-1}V_k(X_u)\right)^i \left(\mathbf{J}_v^{-1}V_k(X_v)\right)^j |u-v|^{2H-2} dudv.$$

Since $\mathbf{D}_s^k X_t = \mathbf{J}_t\mathbf{J}_s^{-1}V_k(X_s)$, we obtain

$$\Gamma(t,x) = \mathbf{J}_t \gamma(t,x) \mathbf{J}_t^T. \tag{3.14}$$

Recall

$$M_{i,j}(t,x) = \frac{1}{t^{2H}} \langle h_i(t,x), h_j(t,x) \rangle_{\mathcal{H}},$$

where

$$h_i(t,x) = (\beta_i^k(s,x)\mathbb{I}_{[0,t]}(s))_{k=1,\dots,n}, \qquad i = 1, ..., n.$$

By (3.8) and (3.14), we have

$$V(x)M(t,x)V(x)^T = \frac{1}{t^{2H}}\gamma(t,x) = \mathbf{J}^{-1}\frac{\Gamma(t,x)}{t^{2H}}(\mathbf{J}^{-1})^T. \tag{3.15}$$

Finally, by equation (3.4), Lemma 2.1 and estimate (3.13) we have for all $p \geq 1$

$$\sup_{t\in[0,1]} \mathbb{E}\left[\det(M_{i,j})^{-p}\right] < \infty,$$

which is the desired result. □

The following definition is inspired by Kusuoka [8].

Definition 3.4. *Let H be a separable real Hilbert space and $r \in \mathbb{R}$ be any real number. Introduce $\mathcal{K}_r(H)$ the set of mappings $\Phi(t,x) : (0,1] \times \mathbb{R}^n \to \mathbb{D}^\infty(H)$ satisfying:*
(1) $\Phi(t,x)$ is smooth in x and $\frac{\partial^\nu \Phi}{\partial x^\nu}(t,x)$ is continues in $(t,x) \in (0,1] \times \mathbb{R}^n$ with probability one for any multi-index ν;
(2) For any $n, p > 1$ we have

$$\sup_{0<t\leq 1} t^{-rH} \left\| \frac{\partial^\nu \Phi}{\partial^\nu x}(t,x) \right\|_{\mathbb{D}^{k,p}(H)} < \infty.$$

We denote $\mathcal{K}_r(\mathbb{R})$ by $\mathcal{K}_r$.

Lemma 3.5. *With probability one, we have*
(1) $\alpha(t,x), \beta(t,x) \in \mathcal{K}_0$;
(2) $\delta h_i(t,x) \in \mathcal{K}_1$;
(3) Let $(M_{i,j}^{-1})$ be the inverse matrix of $(M_{i,j})$. Then $M_{i,j}^{-1} \in \mathcal{K}_0$ for all $i,j = 1, ..., n$.

Proof. The first two statements are immediate consequences of Lemma 3.2. The third statement follows by writing $M^{-1} = \frac{\text{adj} M}{\det M}$, estimates in Lemma 3.2 (1) and Lemma 3.3. □

Now we can state one of our main results in this note.

Theorem 3.6. *Let f be any C^∞ bounded function and $\Phi(t,x) : \Omega \to \mathcal{K}_r$ we have*

$$\mathbb{E}\left(\Phi(t,x)V_i f(X_t^x)\right) = \mathbb{E}\left((T_{V_i}^*\Phi(t,x))f(X_t^x)\right),$$

where $T_{V_i}^*\Phi(t,x)$ is an element in $\mathcal{K}_{r-1}$ with probability one.

Proof. This is primarily integration by parts together with the estimates obtained before. First note

$$\begin{aligned}
\mathbf{D}_s^j f(X_t) &= \langle \nabla f(X_t), \mathbf{D}_s^j X_t \rangle \\
&= \langle \nabla f(X_t), \mathbf{J}_t \mathbf{J}_s^{-1} V_j(X_s) \rangle \\
&= \sum_{k,l=1}^n h_k^j(t) \alpha_l^k(t)(V_l f)(X_t).
\end{aligned}$$

Hence

$$V_i f(X_t) = \frac{1}{t^{2H}} \sum_{j,l=1}^{n} \beta^i_j(t) M^{-1}_{jl} \langle \mathbf{D} f(X_t), h_l(t)\rangle_{\mathcal{H}}. \tag{3.16}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}\left(\Phi(t,x)V_i f(X_t)\right)\\
=&\frac{1}{t^{2H}}\sum_{k,l=1}^{n}\mathbb{E}\left(\langle \mathbf{D}f(X_t), \Phi(t,x)\beta^i_k(t)M^{-1}_{kl}(t)h_l(t)\rangle_{\mathcal{H}}\right)\\
=&\frac{1}{t^{2H}}\sum_{k,l=1}^{n}\mathbb{E}\left(\left[\delta\left(\Phi(t,x)\beta^i_k(t)M^{-1}_{kl}(t)h_l(t)\right)\right]f(X_t)\right)\\
=&\sum_{k,l=1}^{n}\mathbb{E}\left(\left[\frac{1}{t^{2H}}\Phi(t,x)\beta^i_k(t)M^{-1}_{kl}(t)\delta h_l(t) - \frac{1}{t^{2H}}\langle \mathbf{D}(\Phi(t,x)\beta^i_k(t)M^{-1}_{kl}(t)), h_l(t)\rangle_{\mathcal{H}}\right]f(X_t)\right).
\end{aligned}$$

By Lemma 3.5, the first term in the brackets above is in $\mathcal{K}_{r-1}$ and the second term is in $\mathcal{K}_r$. Finally, denote

$$T^*_{V_i}\Phi(t,x) = \sum_{k,l=1}^{n}\left[\frac{1}{t^{2H}}\Phi(t,x)\beta^i_k(t)M^{-1}_{kl}(t)\delta h_l(t) - \frac{1}{t^{2H}}\langle \mathbf{D}(\Phi(t,x)\beta^i_k(t)M^{-1}_{kl}(t)), h_l(t)\rangle_{\mathcal{H}}\right].$$

It is clear that $T^*_{V_i}\Phi(t,x) \in \mathcal{K}_{r-1}$. The proof is completed. □

4. Gradient Bounds

With the integration by parts formula of Theorem 3.6 in hands we can now prove our gradient bounds. We start with the following basic commutation formula:

Lemma 4.1. *For $i = 1, 2, ..., n$, we have the commmutation*

$$V_i P_t f(x) = \mathbb{E}\left(\left((\mathbf{J}_t V_i) f\right)(X^x_t)\right) = \mathbb{E}\left(\sum_{k=1}^{n}\alpha^i_k(t,x)V_k f(X^x_t)\right),$$

where the $\alpha(t,x)$ solve the system of stochastic differential equations (3.9).

Proof. For any C^∞_b-vector field W on $\mathbb{R}^n$ we have

$$WP_t f(x) = \mathbb{E}\left(((\mathbf{J}_t W)f)(X^x_t)\right).$$

The remainder of the proof is then clear from the computations in the previous section. □

Finally we have the following gradient bounds.

Theorem 4.2. *Let $p > 1$. For $i_1, ..., i_k \in \{1, ..., n\}$, and $x \in, \mathbb{R}^n$, we have*

$$|V_{i_1}...V_{i_k}P_t f(x)| \le C(t,x)(P_t f^p(x))^{\frac{1}{p}} \quad t \in [0,1]$$

with $C(t,x) = O\left(\frac{1}{t^{Hk}}\right)$ when $t \to 0$.

Proof. By Theorem 3.6 and Lemma 4.1, for each $k \geq 1$ there exists a $\Phi^{(-k)}(t,x) \in \mathcal{K}_{-k}$ such that

$$V_{i_1}...V_{i_k}P_tf(x) = \mathbb{E}\left(\Phi^{(-k)}(t,x)f(X_t)\right).$$

Now an application of Hölder's inequality gives us the desired result. □

Remark 4.3. *Here let us emphasize a simple but important consequence of the above theorem that, suppose f is uniformly bounded, then*

$$|V_{i_1}...V_{i_k}P_tf(x)| \leq C(t,x)\|f\|_\infty \quad t \in [0,1]$$

where $C(t,x) = O\left(\frac{1}{t^{Hk}}\right)$ as $t \to 0$.

Another direct corollary of Theorem 4.2 is the following inverse Poincaré inequality.

Corollary 4.4. *For $i_1, ..., i_k \in \{1, ..., n\}$, and $x \in \mathbb{R}^n$,*

$$|V_{i_1}...V_{i_k}P_tf(x)|^2 \leq C(t,x)(P_tf^2(x) - (P_tf)^2(x)) \quad t \in [0,1]$$

with $C(t,x) = O\left(\frac{1}{t^{2Hk}}\right)$ when $t \to 0$.

Proof. By Theorem 4.2, for any constant $C \in \mathbb{R}$ we have

$$|V_{i_1}...V_{i_k}P_tf(x)|^2 = |V_{i_1}...V_{i_k}P_t(f-C)(x)|^2 \leq C(t,x)(P_t(f-C)^2(x)) \quad t \in [0,1]$$

with $C(t,x) = O\left(\frac{1}{t^{2Hk}}\right)$ when $t \to 0$. Now minimizing $C \in \mathbb{R}$ gives us the desired result. □

Remark 4.5. *For each smooth function $f : \mathbb{R}^n \to \mathbb{R}$, denote*

$$\Gamma(f) = \sum_{i=1}^n (V_if)^2.$$

We also have, for $i_1, ..., i_k \in \{1, ..., n\}$, and $x \in \mathbb{R}^n$,

$$|V_{i_1}...V_{i_k}P_tf(x)|^2 \leq C(t,x)P_t\Gamma(f)(x), \quad t \in [0,1]$$

with $C(t,x) = O\left(\frac{1}{t^{2H(k-1)}}\right)$ when $t \to 0$. Indeed, by Theorem 3.6 and Lemma 4.1, we know that for each $k \geq 1$, there exists $\Phi_j^{(1-k)}(t,x) \in \mathcal{K}_{1-k}, j = 1, 2, ..., n$ such that

$$V_{i_1}...V_{i_k}P_tf(x) = \mathbb{E}\left(\Phi_j^{(1-k)}(t,x)(V_jf)(X_t)\right).$$

The sequel of the argument is then clear.

5. A global gradient bound

Throughout our discussion in this section, we show that under some additional conditions on the vector fields $V_i, ..., V_n$, we are able to obtain

$$\sqrt{\Gamma(P_tf)} \leq P_t(\sqrt{\Gamma(f)}),$$

uniformly in x, where we denoted as above

$$\Gamma(f) = \sum_{i=1}^n (V_if)^2.$$

For this purpose, we need the following additional structure equation imposed on vector fields $V_i, ..., V_d$.

Hypothesis 5.1. *In addition to Hypothesis 2.2, we assume the smooth and bounded functions* ω_{ij}^k *satisfy:*

$$\omega_{ij}^k = -\omega_{ik}^j, \qquad 1 \le i, j, k \le d.$$

Interestingly, such an assumption already appeared in a previous work of the authors [4] where they proved an asymptotic expansion of the density of X_t when $t \to 0$.

Remark 5.2. *In the case of a stochastic differential equation driven by a Brownian motion, the functional operator* P_t *is a diffusion semigroup with infinitesimal generator* $L = \frac{1}{2}\left(\sum_{i=1}^n V_i^2\right)$*. The gradient subcommutation*

$$\sqrt{\Gamma(P_t f)} \le P_t(\sqrt{\Gamma f}),$$

is then known to be equivalent to the fact that the Ricci curvature of the Riemannian geometry given by the vector fields V_i*'s is non negative (see for instance* [1]*).*

The following approximation result, which can be found for instance in [6], will also be used in the sequel:

Proposition 5.3. *For* $m \ge 1$*, let* $B^m = \{B_t^m; t \in [0,1]\}$ *be the sequence of linear interpolations of* B *along the dyadic subdivision of* $[0,1]$ *of mesh* m*; that is if* $t_i^m = i2^{-m}$ *for* $i = 0, ..., 2^m$*; then for* $t \in (t_i^m, t_{i+1}^m]$*,*

$$B_t^m = B_{t_{i^m}} + \frac{t - t_{i^m}}{t_{i+1}^m - t_i^m}(B_{t_{i+1}^m} - B_{t_i^m}).$$

Consider X^m *the solution to equation (2.2) restricted to* $[0,1]$*, where* B *has been replaced by* B^m*. Then almost surely, for any* $\gamma < H$ *and* $t \in [0,1]$ *the following holds true:*

$$\lim_{m\to\infty} \|X^x - X^m\|_\gamma = 0 \tag{5.17}$$

Theorem 5.4. *Recall the definition of* $\alpha(t,x)$ *in (3.6) and*

$$d\alpha_j^i(t,x) = -\sum_{k,l=1}^n \alpha_k^i(t,x)\omega_{lk}^j(X_t^x)dB_t^l, \qquad \text{with} \quad \alpha_j^i(0,x) = \delta_j^i. \tag{5.18}$$

Under Assumption 5.1, uniformly in t *and* x*, we have*

$$\left\|\sum_{j=1}^n \alpha_j^i(t,x)^2\right\|_{L^\infty} \le 1; \qquad \text{and} \qquad \left\|\sum_{i=1}^n \alpha_j^i(t,x)^2\right\|_{L^\infty} \le 1, \tag{5.19}$$

almost surely.

Proof. Let us thus consider X_t^m and $\alpha^m(t,x)$ the solution of (2.2) and (5.18) where B is replaced by B^m, that is

$$dX_t^m = \sum_{i=1}^n V_i(X_s^m)dB_s^{m,i},$$

$$d\alpha^m(t,x) = -\sum_{k=1}^n \alpha^m(t,x)\omega_k(X_s^m)dB_s^{m,k},$$

with $X_0 = x$ and $\alpha(0,x) = I$. Here $\omega_k = (\omega^i_{kj})$. In order to show that the process $\alpha(t,x)$ is uniformly bounded, by applying Proposition 5.3 to the couple (X,α), it is sufficient to prove our uniform bounds on $\alpha^m(t,x)$, uniformly in m. In the sequel set

$$\Delta B^{k,m}_{t^m_{n-1}t^m_n} := \frac{B^{k,m}_{t^m_n} - B^{k,m}_{t^m_{n-1}}}{t^m_n - t^m_{n-1}}, \quad \text{for} \quad 1 \le n \le 2^m \text{ and } 1 \le k \le n.$$

Then, for $t \in [t^m_{n-1}, t^m_n)$, we have

$$d\alpha^m(t,x) = -\alpha^m(t,x) \sum_{k=1}^n \omega_k(X^m_t)\, \Delta B^{k,m}_{t^m_{n-1}t^m_n} dt,$$

Therefore, for $t \in [t^m_{n-1}, t^m_n)$, we obtain

$$\alpha^m(t,x) = \left(e^{-\sum_{k=1}^n \Delta B^{k,m}_{t^m_{n-1}t^m_n} \int_{t^m_{n-1}}^t \omega_k(X^m_s) ds} \right) \alpha^m(t^m_{n-1}, x)$$

Proceeding inductively, we end up with the following identity, valid for $t \in [t^m_{n-1}, t^m_n)$ and $n = 0, ..., 2^m$:

$$\alpha^m(t,x) \quad = \quad e^{-\sum_{k=1}^n \Delta B^{k,m}_{t^m_{n-1}t^m_n} \int_0^t \omega_k(X^m_s) ds} \quad \times \ \cdots \ \times \ e^{-\sum_{k=1}^n \Delta B^{k,m}_{t^m_0 t^m_1} \int_{t^m_0}^{t^m_1} \omega_k(X^m_s) ds}. \tag{5.20}$$

By Assumption 5.1, each ω_k is a skew-symmetric matrix, expression (5.20) gives us

$$\left\| \sum_{j=1}^n \alpha^{m,i}_j(t,x)^2 \right\|_{L^\infty} \le 1; \qquad \text{and} \quad \left\| \sum_{i=1}^n \alpha^{m,i}_j(t,x)^2 \right\|_{L^\infty} \le 1.$$

This is our claimed uniform bound on $\alpha^m(t,x)$, from which the end of our proof is easily deduced.

□

As a direct consequence of Lemma 4.1 and Theorem 5.4, we have the main result of this section.

Theorem 5.5. *Under Assumption 5.1, we have uniformly in x*

$$\sqrt{\Gamma(P_t f)} \le P_t(\sqrt{\Gamma(f)}).$$

Proof. By applying Lemma 4.1, Cauchy-Schwarz inequality and then Theorem 5.4, we have for any vector $a = (a_i) \in \mathbb{R}^n$

$$\begin{aligned} \sum_{i=1}^n a_i V_i P_t f(x) &= \mathbb{E}\left(\sum_{i,k=1}^n a_i \alpha^i_k(t,x) V_k f(X^x_t) \right) \\ &\le \mathbb{E}\left[\left(\sum_{k=1}^n \left(\sum_{i=1}^n a_i \alpha^i_k(t,x) \right)^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n (V_k f(X^x_t))^2 \right)^{\frac{1}{2}} \right] \\ &\le \|a\| \mathbb{E}\left(\sum_{k=1}^n V_k f(X^x_t)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By choosing

$$a_i = \frac{(V_i P_t f)(x)}{\sqrt{\sum_{i=1}^n (V_i P_t f)^2(x)}},$$

we obtain

$$\sqrt{\Gamma(P_t f)} = \sqrt{\sum_{i=1}^n (V_i P_t f)^2} \le P_t(\sqrt{\Gamma(f)}).$$

The proof is completed. □

Remark 5.6. *Since P_t comes from probability measure, we observe from Jensen inequality that*

$$\sqrt{\Gamma(P_t f)} \le P_t(\sqrt{\Gamma(f)})$$

implies

$$\Gamma(P_t f) \le P_t(\Gamma f).$$

References

[1] D. Bakry, *L'hypercontractivité et son utilisation en théorie des semigroupes*, Ecole d'Eté de Probabilites de St-Flour, Lecture Notes in Math, (1994).

[2] F. Baudoin, L. Coutin: *Operators associated with a stochastic differential equation driven by fractional Brownian motions*, Stochastic Processes and their Applications, **117**, Issue 5, pp. 550-574, (2007).

[3] F. Baudoin, M. Hairer: *A version of Hörmander's theorem for the fractional Brownian motion*, Prob. The. Rel. Fields., **139**, 373-395, (2007).

[4] F. Baudoin, C. Ouyang: *Small-time kernel expansion for solutions of stochastic differential equations driven by fractional Brownian motions*, Stochastic Processes and their Applications **121**, Issue 4, 2011, Pages 759-792

[5] L. Decreusefond and A. S. Üstünel, *Stochastic analysis of the fractional Brownian motion*, Potential Analysis. **10**, 177-214, (1998).

[6] P. Friz, N. Victoir: *Multidimensional dimensional processes seen as rough paths.* Cambridge University Press (2010).

[7] Y. Hu and D. Nualart: *Differential equations driven by Holder continuous functions of order greater than 1/2.* In: Stochastic Analysis and Applications, 399-413, Abel Symp., 2, Springer, Berlin, 2007.

[8] S. Kusuoka: *Malliavin calculus revisited.* J. Math. Sci. Univ. Tokyo, **10** (2003), 261-277.

[9] D. Nualart: *The Malliavin Calculus and Related Topics.* Probability and its Applications. Springer-Verlag, 2nd Edition, (2006).

[10] D. Nualart, A. Rascanu, *Differential equations driven by fractional Brownian motion*, Collect. Math. **53** (1) (2002) 28 5581. 29

[11] D. Nualart, B. Saussereau: *Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion*, Stochastic Processes and their Applications, **119**, Issue 2, 2009, Pages 391-409

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